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EXTENSION OF OVSYANNIKOV'S ANALYTICAL SOLUTIONS TO TRANSONIC FLOWS

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The solution of the equation of the velocity potential of a steady axisymmetric ideal-gas flow in the neighborhood of a given point at the axis of symmetry in the form of a double series in powers of the distance to the axis of symmetry and its logarithm is considered. Recurrent chains of equations with arbitrariness in two analytical functions of the streamwise variable are obtained for coefficients of the series. Convergence of the constructed series is proved by the method of special majorants. The theorem of existence and uniqueness of the solution of the initial-boundary problem for this nonlinear differential equation in partial derivatives with a singularity at the axis of symmetry is obtained as an analog of Kovalevskaya's and Ovsyannikov's theorems.

Key words: *transonic flow, gas dynamics, nozzle, Ovsyannikov's theorem, Kovalevskaya's theorem, series, convergence.*

Introduction. Ovsyannikov [1] proved an analog of Kovalevskaya's theorem, which justifies the use of series in powers of the distance to the axis of symmetry in the inverse problem of gas flows in axisymmetric nozzles. The arbitrariness of the solution of this problem is one analytical function of the streamwise variable (gas velocity at the axis of symmetry). As the equation considered is a second-order equation, the general solution should contain another arbitrary function, which could also be prescribed at the nozzle centerline. Ovsyannikov assumed that this hypothetical function was zero to avoid the singularity at the centerline and consider the neighborhood of the sonic line. In this case, the sought potential of the moving gas is the solution of the characteristic Cauchy problem; the theory of its solution in the form of series in powers of the characteristic variable (in our case, r) was also developed in [2–6, 8].

Titov [9] considered the solution of an axisymmetric problem of a steady transonic flow around slender bodies with the use of logarithmic series. This method is used in the present paper to solve the problem of local construction of solutions of the equation for the velocity potential φ of steady motion of an ideal gas. The solution is constructed in the neighborhood of the axis of symmetry of the flow, because the standard formulation implies solving the inverse problem of nozzle flows [10], where the gas velocity is set at the nozzle centerline and then the flow parameters and the nozzle shape are determined. The solution of the equation is constructed in the form of a double series in power of r and $\ln r$ (or in fractional powers of r) in the neighborhood $r = 0$, $z = z_0$ [7, 10, 11]. Recurrent chains of equations are obtained for coefficients of the series. The convergence of the resultant series is proved by the method of special majorants. The theorem of existence and uniqueness of the solution for this nonlinear differential equation in partial derivatives with a singularity is obtained [11]. If $\varphi_r = 0$ for $r = 0$, then the solution is an analytical function represented as a series in powers of the characteristic variable r [8]; the convergence of this series can be proved by Ovsyannikov's method [1, 10]. If $\varphi_r \neq 0$ for $r = 0$, then non-analytical terms containing $\ln r$ and a second arbitrary function appear in the series [11]. From the mathematical viewpoint, it is necessary to find the second arbitrary function of the streamwise variable. Indeed, if we introduce the second arbitrary function at the axis of symmetry, which "governs" the nonzero transverse derivative φ_r as $r \to 0$, then,

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according to Ovsyannikov's theorem [1], a singularity should arise at the axis of symmetry ($r = 0$): the derivative φ_r turns to infinity, in contrast to the solution of the standard characteristic Cauchy problem [3, 7, 8]. The singularity is identified by determining such a power of r for which the derivative multiplied by this power would not tend to infinity. In the present work, we found a value of ε, such that

$$\lim_{r \to 0} r^{\varepsilon} \varphi_r = \bar{\alpha}(z),$$

where $\bar{\alpha}(z)$ is an arbitrary analytical function of the variable z. In the case considered, we obtain $\varepsilon = (\gamma - 1)/(\gamma + 1)$, which is justified by proving the convergence of the asymptotic series constructed. The potential φ remains an analytical function of the variables r and z for small r other than zero. The problem posed can be called the generalized Cauchy problem. It turned out that this generalized Cauchy problem in our case has a solution, namely, for small r, the function $\varphi(r, z)$ can be asymptotically presented as

$$\varphi \sim \varphi_0 = \alpha(z) r^{\delta} + \beta(z),$$

where $\delta = 2/(\gamma + 1)$. Hence, $\bar{\alpha}(z) = \delta \alpha(z)$.

The solution φ_0 found can be naturally called the generalized initial data for the Cauchy problem being solved. Moreover, it turned out that φ_0 can be considered as the zero coefficient of the logarithmic series converging for small positive r. It should be noted that the arbitrariness of functions being prescribed determines a whole class of analytical solutions of the equation considered. It is this phenomenon that is described in the present paper.

Formulation of the Problem. The equation for the velocity potential Φ of steady motion of a polytropic gas has the form

$$\sum_{ik} (1 - \delta_{ik}) \Phi_{x_i} \Phi_{x_k} \Phi_{x_i x_k} - \sum_{i} (\Theta - \Phi_{x_i}^2) \Phi_{x_i x_i} = 0. \tag{1}$$

Here $\Phi = \Phi(x_1, x_2, x_3)$ (x_i are the Cartesian coordinates) and

$$\Theta = (\gamma - 1)\Big(K - \frac{1}{2} \sum_{i} \Phi_{x_i}^2\Big), \tag{2}$$

where Θ is the velocity of sound squared, $K > 0$, $K = \text{const}$, and γ is the ratio of specific heats; $i, k = 1, 2, 3$.

In the axisymmetric case, we have

$$x_3 = z, \qquad r = \sqrt{x_1^2 + x_2^2}, \qquad \Phi = \Phi(r, z). \tag{3}$$

From Eqs. (1) and (2), we derive the equation

$$2\Phi_r \Phi_z \Phi_{rz} + (\Phi_z^2 - \Theta)\Phi_{zz} - \Theta(\Phi_{rr} + \Phi_r/r) + \Phi_r^2 \Phi_{rr} = 0, \tag{4}$$

and then, from Eqs. (2), (3), we obtain

$$\Theta = (\gamma - 1)[K - \Phi_r^2/2 - \Phi_z^2/2]. \tag{5}$$

Construction of the Logarithmic Series. Now we construct the solution Φ of Eq. (4) in the form of a series

$$\Phi = \varphi(\rho, r, z), \qquad \rho = \ln r; \tag{6}$$

$$\varphi = \sum_{n=0}^{\infty} \varphi_n(\rho, z) r^n. \tag{7}$$

By virtue of Eq. (6), Eqs. (4) and (5) are transformed to

$$2\varphi_z(\varphi_\rho/r + \varphi_r)(\varphi_{\rho z}/r + \varphi_{rz}) + \varphi_{zz}[(\gamma + 1)\varphi_z^2/2 + (\gamma - 1)(\varphi_\rho/r + \varphi_r)^2/2 - (\gamma - 1)K]$$

$$+ (\gamma - 1)[-K + \varphi_z^2/2 + (\varphi_\rho/r + \varphi_r)^2/2][\varphi_{rr} + 2\varphi_{r\rho}/r + \varphi_r/r + \varphi_{\rho\rho}/r^2]$$

$$+ (\varphi_\rho/r + \varphi_r)^2[\varphi_{rr} + 2\varphi_{r\rho}/r + (\varphi_{\rho\rho} - \varphi_\rho)/r^2] = 0. \tag{8}$$

Multiplying Eq. (8) by r^4, we transform this equation to

$$(r\varphi_r+\varphi_\rho)^2[(\gamma+1)(r^2\varphi_{rr}+2r\varphi_{r\rho}+\varphi_{\rho\rho})/2+(\gamma-1)r\varphi_r/2-\varphi_\rho]$$
$$=(\gamma-1)(K-\varphi_z^2/2)r^2[r^2\varphi_{rr}+2r\varphi_{r\rho}+r\varphi_r+\varphi_{\rho\rho}]-2r^2\varphi_z(r\varphi_r+\varphi_\rho)(r\varphi_{rz}+\varphi_{\rho z})$$
$$-r^2\varphi_{zz}[(\gamma+1)\varphi_z^2r^2/2-(\gamma-1)r^2K+(\gamma-1)(r\varphi_r+\varphi_\rho)^2/2]. \tag{9}$$

Substituting Eqs. (6) and (7) into Eq. (9), we obtain the following equation for $n=0$ (the prime indicates the derivative with respect to ρ):

$$(\varphi_0')^2[(\gamma+1)\varphi_0''/2-\varphi_0']=0. \tag{10}$$

If $\varphi_0'=0$ in Eq. (10), we have a traditional power series for the potential; the convergence of this series in the neighborhood of the centerline was proved in Ovsyannikov's classical paper [1]. However, if $\varphi_0'\neq 0$ in Eq. (10), the equality of the second term to zero yields

$$\varphi_0'=\mathrm{e}^{2\rho/(\gamma+1)}\,\tilde{\alpha}(z),$$
$$\varphi_0(\rho,z)=\alpha(z)\,\mathrm{e}^{2\rho/(\gamma+1)}+\beta(z)=\alpha(z)r^{2/(\gamma+1)}+\beta(z). \tag{11}$$

Thus, the order of the zero approximation in Eq. (7) depends on the ratio of specific heats γ:

$$\varphi_0=\alpha(z)r^{3/4}+\beta(z)\qquad\text{for}\quad\gamma=5/3; \tag{12}$$

$$\varphi_0=\alpha(z)r^{5/6}+\beta(z)\qquad\text{for}\quad\gamma=7/5; \tag{13}$$

$$\varphi_0=\alpha(z)\sqrt{r}+\beta(z)\qquad\text{for}\quad\gamma=3; \tag{14}$$

in all cases with $\gamma>1$ and $\alpha(z)=0$, the value of gas velocity at the axis of symmetry $r=0$ coincides with β'.

Collecting formally terms at r^n with $n>0$, after substituting series (6), (7) into (9), we obtain the nth equation of the system for the series coefficients

$$2\varphi_0'\varphi_n\Big[\frac{\gamma+1}{2}\,\varphi_0''-\varphi_0'\Big]+(\varphi_0')^2\Big[\frac{\gamma+1}{2}\,(n(n-1)\varphi_n+2n\varphi_n'+\varphi_n'')$$
$$+\frac{\gamma-1}{2}\,n\varphi_n-\varphi_n'\Big]=F_n(z,\rho), \tag{15}$$

where

$$F_n(z,\rho)=-\sum_{\substack{k+l+m=n\\0\le k,l,m<n}}(k\varphi_k+\varphi_k')(l\varphi_l+\varphi_l')$$
$$\times\Big[\frac{\gamma+1}{2}\,(m(m-1)\varphi_m+2m\varphi_m'+\varphi_m'')+\frac{\gamma-1}{2}\,m\varphi_m-\varphi_m'\Big]$$
$$+(\gamma-1)K[p^2\varphi_p+2p\varphi_p'+\varphi_p'']-\frac{\gamma-1}{2}\sum_{k+l+m=p}\dot\varphi_k\dot\varphi_l[m^2\varphi_m+2m\varphi_m'+\varphi_m'']$$
$$-2\sum_{k+l+m=p}\dot\varphi_k(l\varphi_l+\varphi_l')(m\dot\varphi_m+\dot\varphi_m')+(\gamma-1)K\ddot\varphi_{n-4}$$
$$-\sum_{k+l+m=p}\ddot\varphi_k\Big[\frac{\gamma+1}{2}\,\dot\varphi_{l-1}\dot\varphi_{m-1}+\frac{\gamma-1}{2}\,(l\varphi_l+\varphi_l')(m\varphi_m+\varphi_m')\Big]. \tag{16}$$

Here $p=n-2$ and the dot over the symbol indicates the derivative with respect to z.

Equation (15) can be rewritten as

$$F_n(\varphi_0,\varphi_1,\ldots,\varphi_{n-1})=\frac{\gamma+1}{2}\,(\varphi_0')^2\Big[\varphi_n''+\Big(2n-\frac{2}{\gamma+1}\Big)\varphi_n'+n\Big(n-\frac{2}{\gamma+1}\Big)\varphi_n\Big]. \tag{17}$$

Hence, for $\varphi_0' \neq 0$, we have

$$\varphi_n = \frac{\gamma+1}{2}\, \mathrm{e}^{-n\rho} \int \mathrm{e}^{n\rho}\, G_n(\rho, z)\, d\rho - \frac{\gamma+1}{2}\, \mathrm{e}^{-(n-2/(\gamma+1))\rho} \int \mathrm{e}^{(n-2/(\gamma+1))\rho}\, G_n(\rho, z)\, d\rho, \tag{18}$$

where

$$G_n(\rho, z) = \frac{2}{\gamma+1}\, (\varphi_0')^{-2} F_n(\varphi_0, \varphi_1, \ldots, \varphi_{n-1}) = \frac{\gamma+1}{2}\, \frac{\mathrm{e}^{-4\rho/(\gamma+1)}}{\alpha^2(z)}\, F_n(\rho, z). \tag{19}$$

From Eqs. (11), (15), (18), and (19), we obtain the final expression for the solution φ_n of Eq. (17)

$$\varphi_n(\rho, z) = \frac{(\gamma+1)^2}{4\alpha^2(z)} \Big[\mathrm{e}^{-n\rho} \int\limits_{-\infty}^{\rho} F_n(\tau, z)\, \mathrm{e}^{(n-4/(\gamma+1))\tau}\, d\tau - \mathrm{e}^{-(n-2/(\gamma+1))\rho} \int\limits_{-\infty}^{\rho} F_n(\tau, z)\, \mathrm{e}^{(n-6/(\gamma+1))\tau}\, d\tau \Big] \tag{20}$$

without appearance of inessential constants of integration and with convergence of both integrals. This can be explained as follows: if, in each φ_n, we identify an arbitrary constant

$$C_n(z)\, \mathrm{e}^{-n\rho} - C_n(z)\, \mathrm{e}^{-(n-2/(\gamma+1))\rho} = C_n(z)\, \mathrm{e}^{-n\rho}(1 - \mathrm{e}^{2\rho/(\gamma+1)}),$$

then, the solution acquires an arbitrary term in the form of a series

$$\sum_{n=0}^{\infty} C_n(z)\, \mathrm{e}^{-n\rho}(1 - \mathrm{e}^{2\rho/(\gamma+1)}) r^n = \sum_{n=0}^{\infty} C_n(z) r^{-n} (1 - \mathrm{e}^{2\rho/(\gamma+1)}) r^n = \sum_{n=0}^{\infty} C_n(z) (1 - \mathrm{e}^{2\rho/(\gamma+1)}) = (1 - \mathrm{e}^{2\rho/(\gamma+1)}) C(z).$$

This circumstance can be considered as addition to φ_0 of an arbitrary function $(1 - \mathrm{e}^{2\rho/(\gamma+1)})C(z)$, which can be considered as inessential, because the term $C(z)$ is included into the function $\beta(z)$, and $(-1)\, \mathrm{e}^{2\rho/(\gamma+1)}\, C(z)$ is included into the function $\alpha(z)$.

Structure of Coefficients. With the use of formulas (15), (16), and (20), we can now refine the orders of the terms of series (7). Namely, by induction with allowance for (11), we obtain that, for $0 \leqslant k < n$, the maximum power of r included into φ_k, φ_k', and φ_k'', which is also the index ρ in the exponent, equals γ_k. With allowance for Eqs. (16) and (20), we can consider γ_k as an arithmetic progression, $\gamma_k = k\delta + 2/(\gamma+1)$. Then, by means of recurrent estimates, we find that the maximum power of r included into φ_n is $\gamma_n = n\delta + 2/(\gamma+1)$. No constraints are imposed on the value of δ; therefore, with allowance for Eqs. (15) and (20), we can assume that $\delta = 2\gamma_0$. Consecutively analyzing the coefficients in Eqs. (7) and (15), we can state that $\varphi_n = 0$ for odd n, and series (6), (7) is written in a particular form for even n as

$$\varphi(r, z) = \sum_{n=0}^{\infty} r^{2n} \sum_{k=-1}^{l} \lambda_{n,k}(z)\, \mathrm{e}^{-k\delta\rho},$$

where

$$l = \begin{cases} n-1 & \text{for} \quad n \geq 1, \\ 0 & \text{for} \quad n = 0, \end{cases} \qquad \delta = \frac{2}{\gamma+1}, \tag{21}$$

or

$$\varphi_n = \lambda_{n,-1}(z)\, \mathrm{e}^{\delta\rho} + \lambda_{n,0}(z)\, \mathrm{e}^{0\cdot\delta\rho} + \sum_{k=1}^{n-1} \lambda_{n,k}(z)\, \mathrm{e}^{-k\delta\rho}.$$

If the coefficients $\varphi_n(z, \rho)$ are consecutively calculated by Eqs. (16) and (20), the formulas rapidly become very cumbersome with increasing n:

$$n = 0, \qquad \varphi_0(\rho, z) = \alpha(z)\, \mathrm{e}^{2\rho/(\gamma+1)} + \beta(z); \tag{22}$$

$$n = 1, \qquad F_1 = \frac{\gamma+1}{2} (\varphi_0')^2 \Big[\varphi_1'' + \Big(2 - \frac{2}{\gamma+1}\Big)\varphi_1' + \Big(1 - \frac{2}{\gamma+1}\Big)\varphi_1 \Big] = 0.$$

To avoid appearance of new exponents, we assume that

$$\varphi_1 = 0. \tag{23}$$

Transforming Eq. (16), we can write

$$F_n(z,\rho) = -\sum_{\substack{k+l+m=n\\0\le k,l,m<n}} (k\varphi_k+\varphi_k')(l\varphi_l+\varphi_l')\,\frac{\gamma+1}{2}\,(\varphi_m''+(2m-\delta)\varphi_m'+m(m-\delta)\varphi_m)$$

$$+(\gamma-1)K[p^2\varphi_p+2p\varphi_p'+\varphi_p''] - \frac{\gamma-1}{2}\sum_{k+l+m=p}\dot\varphi_k\dot\varphi_l[m^2\varphi_m+2m\varphi_m'+\varphi_m'']$$

$$-2\sum_{k+l+m=p}\dot\varphi_k(l\varphi_l+\varphi_l')(m\dot\varphi_m+\dot\varphi_m')+(\gamma-1)K\ddot\varphi_{n-4}$$

$$-\sum_{k+l+m=p}\ddot\varphi_k\Big[\frac{\gamma+1}{2}\,\dot\varphi_{l-1}\dot\varphi_{m-1}+\frac{\gamma-1}{2}\,(l\varphi_l+\varphi_l')(m\varphi_m+\varphi_m')\Big].$$

Then, for $n=2$, we have

$$F_2=(\gamma-1)K(\varphi_0'')-\frac{\gamma-1}{2}\,\dot\varphi_0\dot\varphi_0\varphi_0''-2\dot\varphi_0\varphi_0'\dot\varphi_0{}'-\ddot\varphi_0\,\frac{\gamma-1}{2}(\varphi_0')^2$$

$$=(\gamma-1)K\alpha\Big(\frac{2}{\gamma+1}\Big)^2 \mathrm{e}^{2\rho/(\gamma+1)}-\frac{\gamma-1}{2}\,\alpha\Big(\frac{2}{\gamma+1}\Big)^2 \mathrm{e}^{2\rho/(\gamma+1)}[\dot\alpha\,\mathrm{e}^{2\rho/(\gamma+1)}+\dot\beta]^2$$

$$-2\dot\alpha\,\frac{2}{\gamma+1}\,\mathrm{e}^{2\rho/(\gamma+1)}\,\frac{2}{\gamma+1}\,\alpha\,\mathrm{e}^{2\rho/(\gamma+1)}[\dot\alpha\,\mathrm{e}^{2\rho/(\gamma+1)}+\dot\beta]$$

$$-[\ddot\alpha\,\mathrm{e}^{2\rho/(\gamma+1)}+\ddot\beta]\frac{\gamma-1}{2}\,\alpha^2\Big(\frac{2}{\gamma+1}\Big)^2\mathrm{e}^{4\rho/(\gamma+1)}$$

$$=\frac{4K(\gamma-1)}{(\gamma+1)^2}\,\alpha\,\mathrm{e}^{\delta\rho}-\frac{2(\gamma-1)}{(\gamma+1)^2}\,\alpha[(\dot\beta)^2\,\mathrm{e}^{\delta\rho}+2\dot\alpha\dot\beta\,\mathrm{e}^{2\delta\rho}+(\dot\alpha)^2\,\mathrm{e}^{3\delta\rho}]$$

$$-\frac{8}{(\gamma+1)^2}\,\alpha\dot\alpha\dot\beta\,\mathrm{e}^{2\delta\rho}-\frac{8}{(\gamma+1)^2}\,\alpha(\dot\alpha)^2\,\mathrm{e}^{3\delta\rho}-\frac{2(\gamma-1)}{(\gamma+1)^2}\,\alpha^2[\ddot\beta\,\mathrm{e}^{2\delta\rho}+\ddot\alpha\,\mathrm{e}^{3\delta\rho}]=\Big[\frac{4K(\gamma-1)}{(\gamma+1)^2}\,\alpha-\frac{2(\gamma-1)}{(\gamma+1)^2}\,\alpha(\dot\beta)^2\Big]\,\mathrm{e}^{\delta\rho}$$

$$-\Big[\frac{4(\gamma-1)}{(\gamma+1)^2}\,\alpha\dot\alpha\dot\beta+\frac{8}{(\gamma+1)^2}\,\alpha\dot\alpha\dot\beta+\frac{2(\gamma-1)}{(\gamma+1)^2}\,\alpha^2\ddot\beta\Big]\,\mathrm{e}^{2\delta\rho}$$

$$-\Big[\frac{2(\gamma-1)}{(\gamma+1)^2}\,\alpha(\dot\alpha)^2+\frac{8}{(\gamma+1)^2}\,\alpha(\dot\alpha)^2+\frac{2(\gamma-1)}{(\gamma+1)^2}\,\alpha^2\ddot\alpha\Big]\,\mathrm{e}^{3\delta\rho}$$

$$=\frac{2}{\gamma+1}\Big(\Big[\frac{2K(\gamma-1)}{(\gamma+1)}\,\alpha-\frac{\gamma-1}{\gamma+1}\,\alpha(\dot\beta)^2\Big]\,\mathrm{e}^{\delta\rho}-\Big[\frac{2(\gamma-1)}{\gamma+1}\,\alpha\dot\alpha\dot\beta+\frac{4}{\gamma+1}\,\alpha\dot\alpha\dot\beta+\frac{\gamma-1}{\gamma+1}\,\alpha^2\ddot\beta\Big]\,\mathrm{e}^{2\delta\rho}$$

$$-\Big[\frac{\gamma-1}{\gamma+1}\,\alpha(\dot\alpha)^2+\frac{4}{\gamma+1}\,\alpha(\dot\alpha)^2+\frac{\gamma-1}{\gamma+1}\,\alpha^2\ddot\alpha\Big]\,\mathrm{e}^{3\delta\rho}\Big)=\frac{2}{\gamma+1}\,[F_{21}\,\mathrm{e}^{\delta\rho}+F_{22}\,\mathrm{e}^{2\delta\rho}+F_{23}\,\mathrm{e}^{3\delta\rho}],$$

where

$$F_{21}=\frac{2K(\gamma-1)}{\gamma+1}\,\alpha-\frac{\gamma-1}{\gamma+1}\,\alpha(\dot\beta)^2;$$

$$F_{22}=\frac{2(\gamma-1)}{\gamma+1}\,\alpha\dot\alpha\dot\beta+\frac{4}{\gamma+1}\,\alpha\dot\alpha\dot\beta+\frac{\gamma-1}{\gamma+1}\,\alpha^2\ddot\beta;$$

$$F_{23}=\frac{\gamma-1}{\gamma+1}\,\alpha(\dot\alpha)^2+\frac{4}{\gamma+1}\,\alpha(\dot\alpha)^2+\frac{\gamma-1}{\gamma+1}\,\alpha^2\ddot\alpha.$$

Then, taking into account Eq. (20), we obtain

$$\varphi_2 = \frac{\gamma+1}{2\alpha^2}\Big(e^{-2\rho} \int\limits_{-\infty}^{\rho} [F_{21}\, e^{(2-\delta)\rho} + F_{22}\, e^{2\rho} + F_{23}\, e^{(2+\delta)\rho}]\, d\rho$$

$$- e^{-(2-\delta)\rho} \int\limits_{-\infty}^{\rho} [F_{21}\, e^{(2-2\delta)\rho} + F_{22}\, e^{(2-\delta)\rho} + F_{23}\, e^{2\rho}]\, d\rho\Big)$$

$$= \frac{\gamma+1}{2\alpha^2}\Big(\Big[\frac{F_{21}}{2-\delta}\ e^{-\delta\rho} + \frac{F_{22}}{2}\ e^{0\rho} + \frac{F_{23}}{2+\delta}\ e^{\delta\rho}\Big] - \Big[\frac{F_{21}}{2(1-\delta)}\ e^{-\delta\rho} + \frac{F_{22}}{2-\delta}\ e^{0\rho} + \frac{F_{23}}{2}\ e^{\delta\rho}\Big]\Big)$$

$$= \frac{1}{2\alpha^2}\Big(\frac{F_{21}\, e^{-\delta\rho}}{(\delta-2)(\delta-1)} + \frac{F_{22}\, e^{0\rho}}{\delta-2} - \frac{F_{23}\, e^{\delta\rho}}{\delta+2}\Big).$$

Convergence of the Logarithmic Series. Let the functions $\alpha(z)$ and $\beta(z)$ be analytical in the neighborhood of the point $z = z_0$. We prove the convergence of series (7) for small r in this neighborhood.

As the functions $\alpha(z)$ and $\beta(z)$ are analytical, there exist positive constants M and R, such that the following inequalities are valid for all $t \geq 0$:

$$\Big|\frac{\partial^t \alpha(z)}{t!\,\partial z^t}\Big|_{z=z_0} \leq \frac{M'}{R^t}, \qquad \Big|\frac{\partial^t \beta(z)}{t!\,\partial z^t}\Big|_{z=z_0} \leq \frac{M'}{R^t}, \qquad \Big|\frac{\partial^{t-q} \alpha^{-2}(z)}{(t-q)!\,\partial z^{t-q}}\Big|_{z=z_0} \leq \frac{M''}{R^t}. \tag{24}$$

We assume that $M = \max(M', 2M', M'')$.

We consider the majorating series for φ_n, in which the absolute values of the coefficients at the exponents are used:

$$\chi_n = \sum_{k=-1}^{l} |\lambda_{n,k}|\, e^{-k\delta\rho}.$$

Here

$$l = \begin{cases} n-1 & \text{for} \quad n \geq 1, \\ 0 & \text{for} \quad n = 0, \end{cases} \qquad \delta = \frac{2}{\gamma+1}.$$

Obviously, we have

$$|\varphi_n| \leq \chi_n. \tag{25}$$

For $n = 0$, we assume that

$$\chi_0^{[t]} = \sup_{-\infty<\tau\leq\rho} \frac{\partial^t \chi_n(z_0,\tau)}{t!\,\partial z^t}.$$

Then, in accordance with Eq. (22), we obtain

$$\frac{\partial^t \chi_0}{\partial z^t} = \frac{\partial^t \alpha(z)}{\partial z^t}\ e^{2\rho/(\gamma+1)} + \frac{\partial^t \beta(z)}{\partial z^t}.$$

Taking into account Eq. (24), we estimate $\chi_0^{[t]}$:

$$\begin{aligned} \chi_0^{[t]} &= \sup_{-\infty<\tau\leq\rho} \Big(\frac{\partial^t \alpha(z)}{t!\,\partial z^t}\ e^{2\rho/(\gamma+1)} + \frac{\partial^t \beta(z)}{t!\,\partial z^t}\Big), \\ \chi_0^{[t]} &\leq \frac{M'}{R^t} \sup_{-\infty<\tau\leq\rho} (e^{2\rho/(\gamma+1)} + 1) \leq \frac{2M'}{R^t} \leq \frac{M}{R^t}. \end{aligned} \tag{26}$$

For $n > 0$, we assume that

$$\chi_n^{[t]}(\rho) = \sup_{-\infty<\tau\leq\rho} \Big(\frac{\partial^t \chi_n(z_0,\tau)}{t!\,\partial z^t}\ e^{(an+b)\tau}\Big). \tag{27}$$

As $\varphi_n(z,\rho)$ is the sum of the powers of the exponents, there exists the upper limit in Eq. (27).

In accordance with Eq. (20), we have

$$\frac{\partial^t \varphi_n}{\partial z^t} = \frac{(\gamma+1)^2}{4}\Big(\mathrm{e}^{-n\rho} \int\limits_{-\infty}^{\rho} \mathrm{e}^{(n-4/(\gamma+1))\tau} \sum_{q=0}^{t} C_t^q \, \frac{\partial^q F_n(z_0,\tau)}{\partial z^q} \, \frac{\partial^{t-q}}{\partial z^{t-q}} \, \alpha^{-2}(z)\, d\tau$$

$$- \mathrm{e}^{-(n-2/(\gamma+1))\rho} \int\limits_{-\infty}^{\rho} \mathrm{e}^{(n-6/(\gamma+1))\tau} \sum_{q=0}^{t} C_t^q \frac{\partial^q F_n(z_0,\tau)}{\partial z^q} \, \frac{\partial^{t-q}}{\partial z^{t-q}} \, \alpha^{-2}(z)\, d\tau\Big). \qquad (28)$$

It follows from Eq. (28) that

$$\frac{\partial^t \varphi_n(z_0,\rho)}{t!\,\partial z^t} \, \mathrm{e}^{(an+b)\rho} = \frac{(\gamma+1)^2}{4}\Big(\mathrm{e}^{-[n(1-a)-b]\rho} \int\limits_{-\infty}^{\rho} \mathrm{e}^{[n(1-a)-4/(\gamma+1)-3b]\tau}$$

$$\times \sum_{q=0}^{t} C_t^q \, \mathrm{e}^{(an+3b)\tau} \, \frac{\partial^q F_n(z_0,\tau)}{\partial z^q} \, \frac{\partial^{t-q}}{\partial z^{t-q}} \, \alpha^{-2}(z)\, d\tau$$

$$- \mathrm{e}^{-[n(1-a)-b-2/(\gamma+1)]\rho} \int\limits_{-\infty}^{\rho} \mathrm{e}^{[n(1-a)-6/(\gamma+1)-3b]\tau} \sum_{q=0}^{t} C_t^q \, \mathrm{e}^{(an+3b)\tau} \, \frac{\partial^q F_n(z_0,\tau)}{\partial z^q} \, \frac{\partial^{t-q}}{\partial z^{t-q}} \, \alpha^{-2}(z)\, d\tau\Big)$$

$$= \frac{(\gamma+1)^2}{4} \int\limits_{-\infty}^{\rho} (\mathrm{e}^{-[n(1-a)-b]\rho+[n(1-a)-4/(\gamma+1)-3b]\tau} - \mathrm{e}^{-[n(1-a)-b-2/(\gamma+1)]\rho+[n(1-a)-6/(\gamma+1)-2b]\tau})$$

$$\times \sum_{q=0}^{t} C_t^q \, \mathrm{e}^{(an+3b)\tau} \, \frac{\partial^q F_n(z_0,\tau)}{\partial z^q} \, \frac{\partial^{t-q}}{\partial z^{t-q}} \, \alpha^{-2}(z)\, d\tau.$$

Estimating the last equality with allowance for Eqs. (24) and (27), we can use a component-by-component transformation to obtain an inequality where X_n is obtained by substituting χ_n into F_n instead of the corresponding φ_n:

$$\frac{\partial^t \chi_n(z_0,\rho)}{t!\partial z^t} \, \mathrm{e}^{(an+b)\rho} \le \frac{(\gamma+1)^2}{4} \, \frac{M}{R^{t-q}} \sup_{-\infty<\tau\le\rho} \Big(\sum_{q=0}^{t} \frac{\mathrm{e}^{(an+3b)\tau}}{q!} \, \frac{\partial^q X_n(z_0,\tau)}{\partial z^q} \Big)$$

$$\times \Big(\mathrm{e}^{-[n(1-a)-b]\rho} \int\limits_{-\infty}^{\rho} \mathrm{e}^{[n(1-a)-4/(\gamma+1)-3b]\tau} \, d\tau - \mathrm{e}^{-[n(1-a)-b-2/(\gamma+1)]\rho} \int\limits_{-\infty}^{\rho} \mathrm{e}^{[n(1-a)-6/(\gamma+1)-3b]\tau} \, d\tau \Big).$$

Integrating, we obtain

$$\frac{\partial^t \chi_n(z_0,\rho)}{t!\,\partial z^t} \, \mathrm{e}^{(an+b)\rho} \le \frac{(\gamma+1)^2}{4} \, \frac{M}{R^{t-q}} \sup_{-\infty<\tau\le\rho} \Big(\sum_{q=0}^{t} \frac{\mathrm{e}^{(an+3b)\tau}}{q!} \, \frac{\partial^q X_n(z_0,\tau)}{\partial z^q} \Big)$$

$$\times \Big(\frac{\mathrm{e}^{-[n(1-a)-b]\rho}}{n(1-a)-4/(\gamma+1)-3b} \, \mathrm{e}^{[n(1-a)-4/(\gamma+1)-3b]\tau} \Big|_{-\infty}^{\rho}$$

$$- \frac{\mathrm{e}^{-[n(1-a)-b-2/(\gamma+1)]\rho}}{n(1-a)-6/(\gamma+1)-3b} \, \mathrm{e}^{[n(1-a)-6/(\gamma+1)-3b]\tau} \Big|_{-\infty}^{\rho} \Big) = \frac{(\gamma+1)^2}{4} \, \frac{M}{R^{t-q}} \, \mathrm{e}^{-(4/(\gamma+1)+2b)\rho} \, \frac{2}{\gamma+1}$$

$$\times \frac{1}{[n(1-a)-4/(\gamma+1)-3b][n(1-a)-6/(\gamma+1)-3b]} \sup_{-\infty<\tau\le\rho} \Big(\sum_{q=0}^{t} \frac{\mathrm{e}^{(an+3b)\tau}}{q!} \, \frac{\partial^q X_n(z_0,\tau)}{\partial z^q} \Big).$$

Using the property of monotonicity of the right side, we take the upper limit in terms of $\rho \le \rho_0 \le 0$ in both sides of the last inequality and write it as

$$\chi_n^{[t]}(\rho_0) \le \frac{(\gamma+1)M}{2R^{t-q}} \frac{1}{[n(1-a)-4/(\gamma+1)-3b][n(1-a)-6/(\gamma+1)-3b]} \sup_{-\infty<\tau\le\rho} \Big(\sum_{q=0}^{t} \frac{\mathrm{e}^{(an+3b)\tau}}{q!} \frac{\partial^q X_n(z_0,\tau)}{\partial z^q}\Big). \quad (29)$$

The form of the dependence $\chi_n(\rho)$ proves the validity of the estimate

$$|\chi_n'(\rho)| \le n|\chi_n|,$$

because $2(n-1)/(\gamma+1) \le n$, from which we find that the relation $(n(1-\gamma)-2)/(\gamma+1) \le 0$ is valid for all $n \ge 0$ for $\gamma > 1$.

Using the definition of $F_n(z,\rho)$ and differentiating in accordance with Leibnitz's rule, we rewrite Eq. (29) in the following form:

$$\chi_n^{[t]}(\rho_0) \le \frac{(\gamma+1)M}{2R^{t-q}} \frac{1}{[n(1-a)-4/(\gamma+1)-3b][n(1-a)-6/(\gamma+1)-3b]}$$

$$\times \Big(8 \sum_{k+l+m=n} \sum_{i=0}^{q} \sum_{j=0}^{i} k\chi_k^{[j]} l\chi_l^{[i-j]} m^2 \chi_m^{[q-i]} + 4(\gamma-1)Kp^2\chi_p^{[q]}$$

$$+ 2(\gamma-1) \sum_{k+l+m=p} \sum_{i=0}^{q} \sum_{j=0}^{i} (j+1)\chi_k^{[j+1]}(i-j+1)\chi_l^{[i-j+1]} m^2 \chi_m^{[q-i]}$$

$$+ 8 \sum_{k+l+m=p} \sum_{i=0}^{q} \sum_{j=0}^{i} (j+1)\chi_k^{[j+1]} l\chi_l^{[i-j]} m(q-i+1)\chi_m^{[q-i+1]} + (\gamma-1)K(q+1)(q+2)\chi_{n-4}^{[q+2]}$$

$$+ \frac{\gamma+1}{2} \sum_{k+l+m=p} \sum_{i=0}^{q} \sum_{j=0}^{i} (j+2)(j+1)\chi_k^{[j+2]}(i-j+1)\chi_{l-1}^{[i-j+1]}(q-i+1)\chi_{m-1}^{[q-i+1]}$$

$$+ 2(\gamma-1) \sum_{k+l+m=p} \sum_{i=0}^{q} \sum_{j=0}^{i} (j+2)(j+1)\chi_k^{[j+2]} l\chi_l^{[i-j]} m\chi_m^{[q-i]}\Big). \quad (30)$$

The transition from (29) to (30) is justified by the estimate of the parameters a and b. Because of the statements above, the numbers a and b should satisfy the conditions

$$b < -\frac{2}{\gamma+1}, \quad 0 < a < 1 - \frac{6}{\gamma+1} - 3b, \quad 0 < a < 1 - \frac{4}{\gamma+1} - 3b,$$

$$a+b>0, \quad 2a+b>0, \quad 1-\frac{4}{\gamma+1}-3b>0, \quad 1-\frac{6}{\gamma+1}-3b>0.$$

We transform this system to

$$b < -\frac{2}{\gamma+1}, \quad 0 < a < 1 - \frac{6}{\gamma+1} - 3b, \quad a+b>0, \quad 1-\frac{6}{\gamma+1}-3b>0.$$

To solve this system, we assume that $b = -2c/(\gamma+1)$, where $c > 1$, and estimate the inequality

$$0 < a < 1 - \frac{6}{\gamma+1} - 3b \quad \Rightarrow \quad 0 < a < 1 - \frac{6}{\gamma+1} + \frac{3\cdot 2c}{\gamma+1} = \frac{6c-5+\gamma}{\gamma+1} = a_{\max}.$$

It follows from here that $(6c-5+\gamma)/(\gamma+1) > 0$ for $c > 1$ and $\gamma > 1$, i.e., this inequality is satisfied. We check satisfaction of the inequality

$$a > -b \quad \Rightarrow \quad a_{\max} > -b \quad \Rightarrow \quad \frac{6c-5+\gamma}{\gamma+1} > \frac{4c}{\gamma+1}.$$

Thus, the inequality is satisfied for $c > 1$ and $\gamma > 1$.

Hence, we can conclude that, for all $b \in (-\infty; -2/(\gamma+1))$, there is $a \in (2c/(\gamma+1); (\gamma-5)/(\gamma+1)-2b)$ such that the system has a solution.

We determine $\psi_n^{[t]}$ by the following recursion:

$$\psi_0^{[t]} = M/R^t \geq \chi_0^{[t]}, \qquad \psi_1^{[t]} = 0,$$

$$\psi_n^{[t]} = \frac{(\gamma+1)M}{2R^{t-q}} \frac{A}{n(n-1)} \Big(8 \sum_{k+l+m=n} \sum_{i=0}^{q} \sum_{j=0}^{i} k\psi_k^{[j]} l\psi_l^{[i-j]} m^2 \psi_m^{[q-i]} + 4(\gamma-1)Kp^2\psi_p^{[q]}$$

$$+ 2(\gamma-1) \sum_{k+l+m=p} \sum_{i=0}^{q} \sum_{j=0}^{i} (j+1)\psi_k^{[j+1]} (i-j+1)\psi_l^{[i-j+1]} m^2 \psi_m^{[q-i]}$$

$$+ 8 \sum_{k+l+m=p} \sum_{i=0}^{q} \sum_{j=0}^{i} (j+1)\psi_k^{[j+1]} l\psi_l^{[i-j]} m(q-i+1)\psi_m^{[q-i+1]} + (\gamma-1)K(q+1)(q+2)\psi_{n-4}^{[q+2]}$$

$$+ \frac{\gamma+1}{2} \sum_{k+l+m=p} \sum_{i=0}^{q} \sum_{j=0}^{i} (j+2)(j+1)\psi_k^{[j+2]} (i-j+1)\psi_{l-1}^{[i-j+1]} (q-i+1)\psi_{m-1}^{[q-i+1]}$$

$$+ 2(\gamma-1) \sum_{k+l+m=p} \sum_{i=0}^{q} \sum_{j=0}^{i} (j+2)(j+1)\psi_k^{[j+2]} l\psi_l^{[i-j]} m\psi_m^{[q-i]} \Big). \tag{31}$$

A positive value of A exists because

$$\lim_{n\to\infty} \frac{n(n-1)}{(n(1-a)-4/(\gamma+1)-3b)(n(1-a)-6/(\gamma+1)-3b)} = \frac{1}{(1-a)^2}.$$

Comparing Eq. (31) with inequalities (27) and (30), and taking into account the equality $\chi_1^{[t]} = 0$, we obtain

$$\chi_n^{[t]}(\rho_0) \leq \psi_n^{[t]};$$

for all n, we have $t \geq 0$ and $\rho_0 \leq 0$, since $\psi_n^{[t]} \geq 0$. Taking into account Eq. (25), we conclude that

$$\varphi_n^{[t]}(\rho_0) \leq \psi_n^{[t]}.$$

We consider the function

$$\Psi(z,y) = \sum_{t=0}^{\infty} \sum_{n=0}^{\infty} \psi_n^{[t]} (z-z_0)^t y^n.$$

According to Eq. (31), this function is the solution of the equation

$$y^2\Psi_{yy} = [(\gamma+1)MA/(2R^{t-q})][8y^2\Psi_y^2(y^2\Psi_{yy} + y\Psi_y)$$

$$+ 4(\gamma-1)Ky^2(y^2\Psi_{yy} + y\Psi_y) + 2(\gamma-1)y^2\Psi_z^2(y^2\Psi_{yy} + y\Psi_y) + 8y^2\Psi_z y\Psi_y$$

$$+ K(\gamma-1)y^4\Psi_{zz} + (\gamma+1)y^2\Psi_{zz}y^2\Psi_z^2/2 + 2(\gamma-1)y^2\Psi_{zz}y^2\Psi_y^2].$$

By cancelling y^2 and explicitly expressing Ψ_{yy}, we can reduce this equality to an equation of Kovalevskaya's type relation

$$\Psi_{yy} = \frac{B(8y\Psi_y^3 + y\Psi_y + 2(\gamma-1)\Psi_z^2 y\Psi_y + 8\Psi_z y\Psi_y y\Psi_{yz})}{1 - B(8y^2\Psi_y^2 + 4(\gamma-1)Ky^2 + 2(\gamma-1)y^2\Psi_z^2)}$$

$$+ \frac{B(K(\gamma-1)y^2\Psi_{zz} + \frac{\gamma+1}{2}\Psi_{zz}\Psi_z^2 + 2(\gamma-1)\Psi_{zz}y^2\Psi_y^2)}{1 - B(8y^2\Psi_y^2 + 4(\gamma-1)Ky^2 + 2(\gamma-1)\Psi_z^2 y^2)}, \tag{32}$$

where $B = (\gamma+1)MA/(2R^{t-q})$, with analytical Cauchy data on the straight line $y = 0$ [in accordance with Eq. (31)]

$$\Psi(z,0) = \frac{MR}{R-(z-z_0)} = \sum_{t=0}^{\infty} \psi_0^{[t]} z^t; \qquad \Psi_y(z,0) = 0. \tag{33}$$

According to Kovalevskaya's theorem, the function $\Psi(z, y)$ from Eqs. (32) and (33) is analytical in a certain neighborhood of the point $z = z_0$, $y = 0$; since its coefficients $\psi_n^{[t]}$ majorate $\varphi_n^{[t]}$, then the series

$$\varphi(z,y) = \sum_{t=0}^{\infty}\sum_{n=0}^{\infty} \varphi_n^{[t]} (z-z_0)^t y^n \tag{34}$$

also converges in this neighborhood.

From Eq. (7) with $\rho = \ln r$, we have

$$r^b \varphi(z,r,\rho) = \sum_{n=0}^{\infty} \Big[\mathrm{e}^{(an+b)\rho} \sum_{t=0}^{\infty} \frac{\partial^t \varphi_n(z_0,\rho)}{t!\,\partial z^t} (z-z_0)^t \Big] r^{n(1-a)}. \tag{35}$$

It follows from Eqs. (26), (27), and (35) that series (7) converges in the region of convergence of series (34), where $y = \sqrt[1-a]{r}$, $r \le \mathrm{e}^{\rho_0} \le 1$. Thus, we proved the following theorem.

Theorem 1. *The generalized Cauchy problem for Eq.* (4) *with the generalized initial data* (11) *has a solution for all functions $\alpha(z)$ and $\beta(z)$ analytical in the neighborhood of the point $z = z_0$ in the form of series* (7) *with coefficients recurrently calculated by formulas* (11), (16), *and* (20) *for all rather small positive r, which is an analytical function in terms of the variables z and r in this neighborhood.*

Physical Meaning. In contrast to [1], where a conventional nozzle is considered, there are singularities in flows at the axis of symmetry $r = 0$; the physical meaning of these singularities requires additional research. In choosing the point $z = z_0$ lying at the line of the transition through the velocity of sound, we obtain a possibility of constructing an analytical solution in both supersonic and subsonic regions. The series constructed can also be applied to solve the inverse problem of external flow around axisymmetric bodies: the point z_0 is chosen inside the body or on its surface, and if the region of convergence of the series is rather large, the gas velocity is calculated in the flow region as well; the shape of the body corresponds to one of the streamlines. This process is similar to replacement of the body by a distribution of singularities of sources at the axis. The method considered can also be used to solve an unsteady potential equation.

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